Review of Lagrangian and Hamiltonian Dynamics in Classical Mechanics
Material from Physics 410

The basic process of Lagrangian mechanics is as follows:

1) Identify all of the constraints on the system.
2) Identify the degrees of freedom of the constrained system. Choose a set of generalized coordinates $(q_1, q_2, \ldots, q_n)$ that incorporate all of the constraints and provide a minimal description (smallest n) of the most general motion of the constrained system.
3) Express the kinetic and potential energy of the system in terms of only the generalized coordinates, their time derivatives, and possibly also time. One can often use a Cartesian, cylindrical or spherical framework in which to write down the components of velocity. This helps when calculating the kinetic energy as $T = \left(\frac{m}{2}\right)\vec{v}\cdot\vec{v}$. Write out the full Lagrangian $\mathcal{L}(q,\dot{q}) = T - U$, where U is the potential energy.
4) Write out and solve the Euler-Lagrange equations for each generalized coordinate. One can think of the Euler-Lagrange equation as "generalized force" = time rate of change of "generalized momentum". $\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}, i = 1, 2, \ldots n.$

The Lagrangian was engineered specifically to reproduce Newton's second law of motion in component form, however it does not have a simple physical interpretation. The Hamiltonian is defined as $\mathcal{H} = \sum_{i=1}^{n} p_i\,\dot{q}_i - \mathcal{L}$, where $p_i = \partial\mathcal{L}/\partial\dot{q}_i$. If there is a time-independent relationship between the Cartesian coordinates and the generalized coordinates, $\vec{r}_\alpha = \vec{r}_\alpha(q_1, q_2, \ldots q_i, \ldots q_n)$, then the Hamiltonian has a simple interpretation as the total mechanical energy $T + U$.

One can solve the n canonical momentum equations $(p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i})$ for $\dot{q}_i$ in terms of the coordinates q_i and momenta p_i to re-express all of the velocities in terms of just the coordinates and momenta $\dot{q}_i = \dot{q}_i(q_i, p_i)$. With this, one can express the Hamiltonian in terms of coordinates and momenta alone $\mathcal{H}(q,p)$, essentially employing a Legendre transformation to move from $(q_i, \dot{q}_i)$ to (q_i, p_i) as the independent variables. Taking the derivative of the Hamiltonian with respect to q_i and p_i, one finds Hamilton's equations: $\dot{q}_i = \partial\mathcal{H}/\partial p_i$ and $\dot{p}_i = -\partial\mathcal{H}/\partial q_i, i = 1, \ldots, n$. This is a set of $2n$ first-order differential equations, as opposed to the set of n second-order differential equations one gets from Lagrange's equations. The Hamiltonian dynamics formulation is useful for quantum mechanics and for classical statistical mechanics. As a way of solving classical mechanics problems it has few advantages over Lagrangian dynamics.

The procedure of utilizing the Hamiltonian method is: (1) choose the generalized coordinates q_i, (2) write down $T, U,$ and $\mathcal{L}$ in terms of the coordinates and their time-derivatives, (3) compute the conjugate momenta $p_i = \partial\mathcal{L}/\partial\dot{q}_i$, (4) express the $\dot{q}_i$ in terms of q_i *and* p_i, (5) compute the Hamiltonian $\mathcal{H}$, and (6) write out and solve Hamilton's equations.

The advantages of the Hamiltonian formulation of mechanics: 1) It is the last step before making the observables into operators and developing quantum mechanics, 2) It allows for a powerful geometrical interpretation of classical mechanics in phase space, 3) and this in turn makes it very useful for nonlinear dynamics and statistical mechanics, 4) it is well-suited for applying perturbation theory, 5) it allows for ignorable coordinates to be fully exploited, and 6) the Hamilton equations are first-order-in-time differential equations, which are often easier to solve than the second-order Euler-Lagrange equations.